Analysis of the Continued Fraction Digits of π

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Background on Continued Fractions

Continued Fraction Expansions

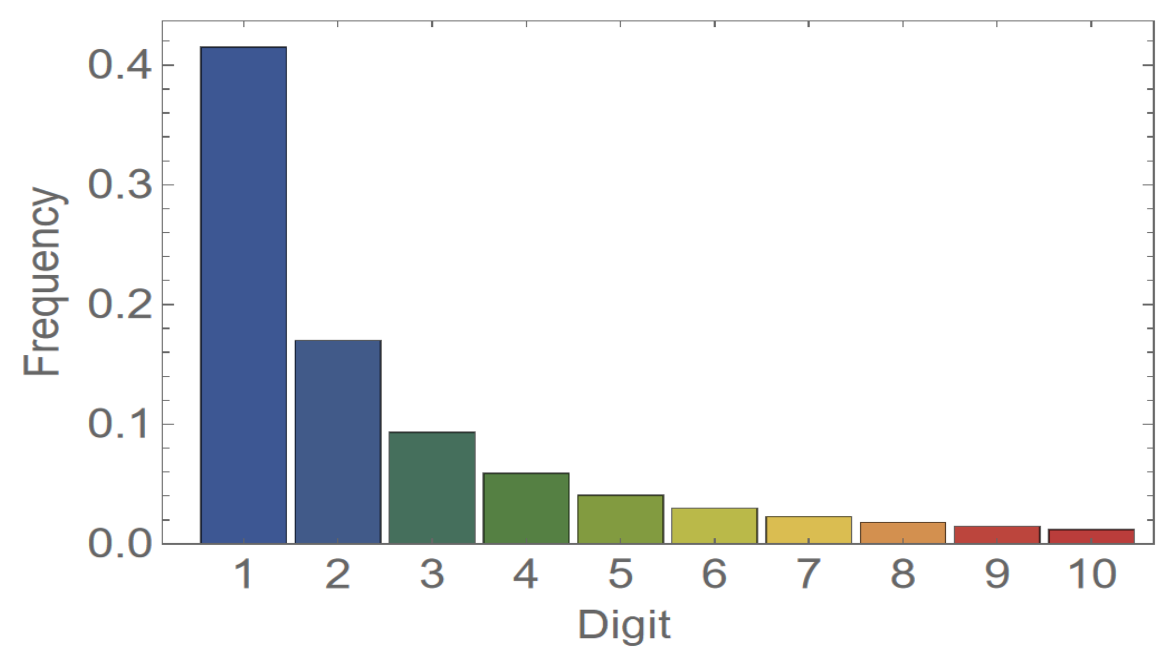
- A **continued fraction** is the representation of a real number x in the form
$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \frac{1}{\cdots}}} = [a_0; a_1, a_2, a_3, \dots]$$
- Continued fraction expansions of irrational numbers are infinite, while rational numbers have terminating expansions.
- **Continued fraction expansion of π:**
$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \frac{1}{\cdots}}} = [3, 7, 15, 1, 292, 1, 1, 1, 2, \dots]$$

Conjecture: The continued fraction digits of π behave like those of a random number $x \in [0,1]$, i.e., they follow the **Gauss-Kuzmin Distribution**, defined on the right.

Statistics of Continued Fraction Digits

Theorem (Gauss-Kuzmin): For almost all real numbers x, the frequency of the digit k in the continued fraction expansion of x is given by

$$\mathbb{P}(k) = \log_2\left(1 + \frac{1}{k(k+2)}\right)$$



Block Frequencies in Continued Fraction Digits

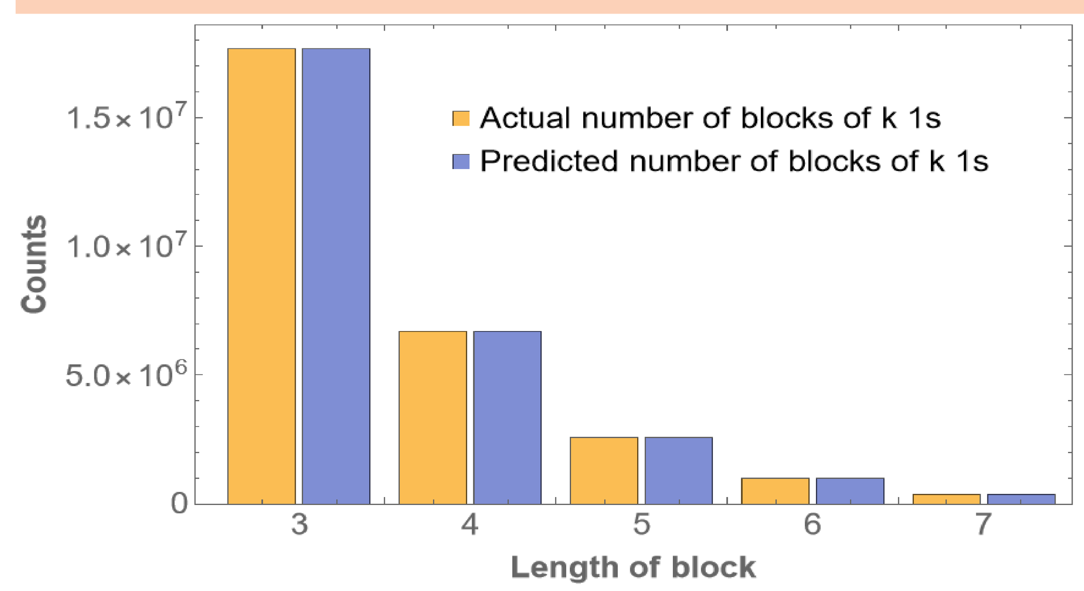
Theoretical Frequencies of Blocks of 1s

Theorem: For almost all real numbers x, the frequency of block of k consecutive 1s in the continued fraction expansion of x is

$$P(\underbrace{1, 1, \cdots, 1}_{k \text{ terms}}) = \left|\log_2\left(1 + \frac{(-1)^k}{F_{k+2}^2}\right)\right|,$$

where F_k is the k-th Fibonacci number.

Observed Frequencies in the first 300,000,000 CF Digits of π



References

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Statistics of Continued Fraction Digits of π

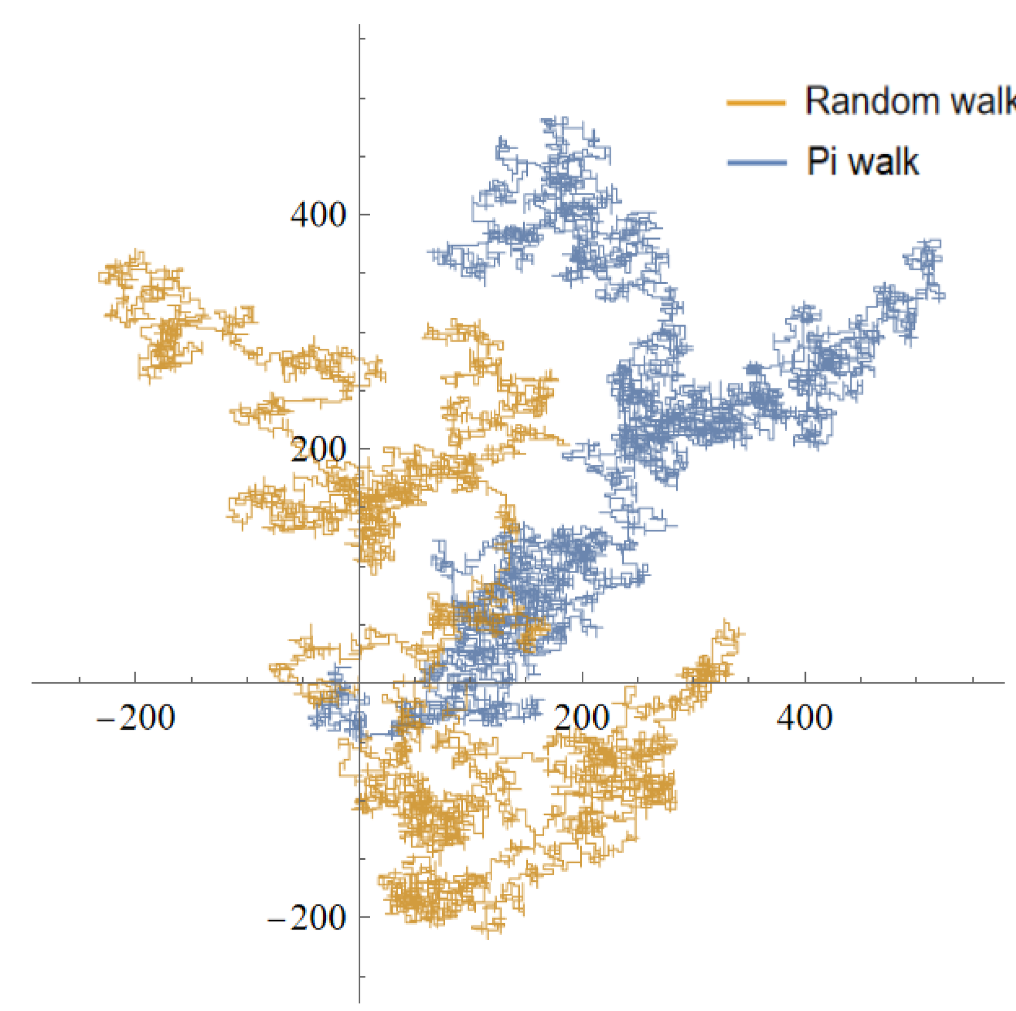
Predicted vs. Actual Digit Counts in the first 30,113,021,586 CF Digits of π

Digit	Predicted Count	Actual Count
1	12,498,033,174.78	12,497,961,253
2	5,116,955,236.43	5,117,043,707
3	2,803,805,504.30	2,803,765,779
4	1,773,466,929.75	1,773,427,556
5	1,223,852,956.47	1,223,886,469

Digit	Predicted Count	Actual Count
6	895,782,393.75	895,746,719
7	684,170,154.08	684,156,432
8	539,682,802.38	539,714,866
9	436,625,855.22	436,649,221
10	360,532,416.93	360,545,777

Random Walks based on Continued Fraction Digits of π

Walk based on the first 10,000 CF Digits mod 4 of π vs. randomly generated walk



Random walk statistics based on the first 1,000 blocks of 1,000,000 CF Digits of π

Statistic	π walk	Random walk
Avg. # of sites visited	147,905.047	146,080.107
Std. Dev.	11,587.097	11,719.355
Avg. Distance to Origin	492.538	499.590
Std. Dev.	202.048	206.720